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I.e., the expressions $\delta^2 e$ and $\delta^2 E$ are connected by the relation

$$\delta^2 E = \frac{(X + \delta X)^2}{X} \delta^2 e. \tag{12}$$

Since the coefficient of $\delta^2 e$ in (12) is positive, it follows that the criterion of stability $\delta^2 E(x_1, x_2, x_3, x_4) \geq 0$ and the criterion $\delta^2 e(y_1, y_2, y_3) \geq 0$ are equivalent. In both forms of the criterion the sign of equality holds when the variations of the variables satisfy the conditions (3). When these conditions are satisfied, e is constant and the equation

$$\delta E = (X + \delta X)\delta^1 e + e\delta X + \frac{(X + \delta X)^2}{X} \delta^2 e + \cdots$$

reduces to $\delta E = e\delta X$, which represents an obvious physical fact. Since the foregoing method of obtaining the results (8) and (12) is tedious, I should be glad to learn of a simpler way of finding them.

III. Graphical Constructions for Imaginary Intersections of Line and Conic.[1]

By R. M. Mathews, University of Minnesota.

Introduction.—Several methods commonly known for the graphical solution of a quadratic equation are incomplete, as they are explained only for real roots. The object of this paper is to complete these constructions and to generalize them to find graphically the intersections of an arbitrary line with any conic.

Intersection of a circle and a line.—When a line l cuts a circle O the length of the chord formed is $2\sqrt{r^2 - d^2}$, where r is the radius of the circle and d is the distance from the center O to the chord.

When the line does not cut the circle in real points, $d > r$, the sign of $r^2 - d^2$ is negative and the length of the chord is imaginary. Let us replace the given circle by a second which cuts the line in real points such that the length of the chord formed is $2\sqrt{d^2 - r^2}$. Draw OM perpendicular to the line to cut the circle at T and l at M, and extend OM to O' so that $MO' = r$ (Fig. 1); with O' as center draw a circle tangent to the given one at T. The length of the chord which it cuts on l is $2\sqrt{d^2 - r^2} = 2i\sqrt{r^2 - d^2}$.

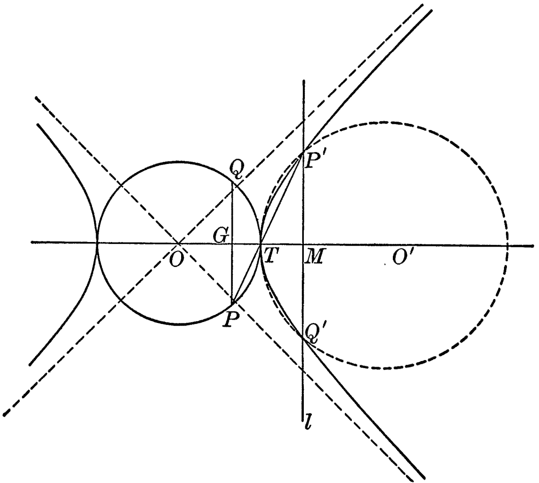

Fig. 1.

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[1] Read before the Minnesota Section of the Mathematical Association of America, May 31, 1919.

If the points of intersection be P' and Q', they may represent the imaginary points in which the line cuts the given circle, and $P'Q'$ the length of the imaginary chord.

Let us call this second circle O' the *virtual image* of the given circle with regard to the line.

Applications in analysis.—(i) The graph of

$$x^2 + y^2 + 2gx + 2fy + k = 0 \tag{1}$$

is a circle with center at $(-g, -f)$ and radius $+\sqrt{g^2 + f^2 - k}$. The graph of

$$\alpha x + \beta y + \gamma = 0 \tag{2}$$

is a straight line with intercepts $(-\gamma/\alpha, 0)$, $(0, -\gamma/\beta)$. The algebraic solution of the system (1), (2) leads to a quadratic equation in x

$$ax^2 + bx + c = 0. \tag{3}$$

whence

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}. \tag{4}$$

The values of y are found by substitution of these x's in (2). Let P and Q be the points of intersection and M the midpoint of the chord PQ.

When the points P and Q are real we ordinarily read out their coördinates directly and so neglect to observe that in (4) the number $-b/2a$ is the abscissa of M while $\sqrt{b^2 - 4ac}/2a$ is half the difference of the abscissæ of P and Q. This guides us in reading out the solutions for the case of imaginary points of intersection. Let the virtual image of (1) cut (2) in points P' and Q', with M' the midpoint of chord. For the intersection of this image with (2), we have

$$x' = \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a},$$

and accordingly the values for x are

abscissa of $M' \pm i/2 \times$ (difference of abscissæ of P' and Q'),

while the values for y are

ordinate of $M' \pm i/2 \times$ (difference of ordinates of P' and Q').

(ii) The algebraic solution of the system of equations

$$x^2 + y^2 + 2g_1x + 2f_1y + k_1 = 0, \tag{5}$$

$$x^2 + y^2 + 2g_2x + 2f_2y + k_2 = 0, \tag{6}$$

is effected by subtracting the one from the other and solving the resulting linear equation with (1). The graph of this linear equation is the radical axis of the two circles given by the equations. Thus the graphical solution consists in constructing the radical axis of the circles and finding its intersections with one of them.

(iii) A combination of the preceding results enables us to construct a length to represent the imaginary tangent from a point to a circle and to find a point whence we can read the complex coördinates of the imaginary points of tangency.

Graphical solution of a quadratic equation.—If x_1 and x_2 are the roots of the quadratic equation

$$x^2 + px + q = 0,$$

then

$$x_1 + x_2 = -p, \qquad x_1 \cdot x_2 = q.$$

Take OM on the x-axis equal to $-p/2$; on the y-axis mark two points H and K such that $OH \cdot OK = q$. Draw that circle through H, K which has its center on the line through M parallel to OY. If this circle cuts OX in x_1 and x_2, then $x_1 \cdot x_2 = OH \cdot OK = q$, and $\frac{1}{2}(x_1 + x_2) = -\frac{1}{2}p$. Thus the abscissæ x_1 and x_2 are roots of the quadratic.

When the circle fails to cut the x-axis, construct its virtual image with regard to the axis. It will cut the line in two points x, and x_2 and the roots of the quadratic are

$$OM \pm \tfrac{1}{2}i \, | x_1 - x_2 | .$$

The proxy of a circle.—To extend these developments to any conic we examine further the geometry of a circle and its virtual image. The point T is a center of similitude of the two circles (Fig. 1). Points on the image can be constructed by drawing any radius OX and then line XT to be cut in X' by a radius $O'T'$ parallel to OX. In particular, on OT take OG as a third proportional to OM and OT. Draw chord through G perpendicular to OT to cut circle at P; then PT cuts l at P' on l.

Secondly, each line of the pencil of parallels to l has a virtual image of the circle O and is cut by it in two points P' and Q'. The locus of P' is an equilateral hyperbola, with center at O, vertex at T, and with OT as transverse axis. Accordingly, each line parallel to l either cuts the circle in real points or cuts the hyperbola in points such that the discriminant for the solution of line and hyperbola is the negative of the discriminant for the solution of line and circle. For this reason it is proposed to call the hyperbola the *proxy* of the circle for the direction l.

Finally, the hyperbola can be plotted from the circle by taking for each line l a corresponding point G and proceeding as explained above.

Intersection of an ellipse and a line.—Consider a system of a circle, a line, the virtual image and the proxy of the circle with regard to the line. Rotate the system around a convenient diameter of the given circle, and project orthogonally upon the initial plane. The two circles become ellipses, the line a line, and the proxy equilateral hyperbola becomes a general hyperbola, because this transformation leaves the line at infinity invariant. For this reason parallel lines go into parallel lines, and similar triangles become similar triangles. Thus, all constructions that depend upon relations of similarity and proportion in the original figure are preserved, with the modification that orthogonal diameters become conjugate diameters.

The details for graphical solution of an ellipse with a line may now be recounted (Fig. 2). Across the curve draw two chords parallel to l and join their midpoints, obtaining the diameter conjugate to l. Let it cut the conic at T' and T, and cut l at M. Bisect $T'T$ for center O. Locate G on OT so that OG is a third proportional to OM and OT; through G draw a chord parallel to l and project its ends through T upon l as P' and Q'. Now the abscissæ of the points of intersection of the line and the ellipse are

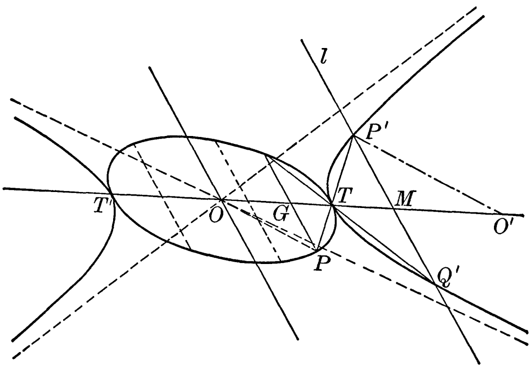

FIG. 2.

abscissa of $M \pm \frac{1}{2}i \times$ (difference of abscissæ of P' and Q'),

and the ordinates are

ordinate of $M \pm \frac{1}{2}i \times$ (difference of ordinates of P' and Q').

The center of the ellipse which is the virtual image of the given one is on OM extended to O' with $MO' = OT$. This image ellipse can be traced by drawing secants through T. If such an one, s, cuts the ellipse at R, draw $O'R'$ parallel to OR to cut s at R', a desired point. Or draw two conjugate diameters through O' to correspond to a convenient pair in the initial curve, and then plot the image with a trammel.

The locus of the points P' and Q' as line l moves parallel to itself is the hyperbola proxy for the ellipse. It can be plotted by locating P' and Q' for l in a series of positions; or we can remark that we have three points, T and one pair P', Q', and a pair of conjugate diameters, whence the curve can be constructed in a variety of ways.

Intersection of an hyperbola and a line.—We have just seen that for an ellipse and a pencil of parallels the proxy is an hyperbola. This hyperbola cuts a line of the pencil in two points such that in the analytic expression for their coördinates the discriminant is the *negative* of the discriminant for said line and the ellipse. Accordingly, those lines which the ellipse cuts in real points the hyperbola cuts in points with the corresponding complex coördinates. Thus the proxy of a hyperbola is that ellipse for which it is proxy, and all the constructions of the preceding paragraph can be applied to the case of a line and hyperbola.

Intersection of a parabola and a line.—The tangent to a parabola at a point T separates all lines parallel to it into two classes, those which cut the curve in real points and those which do not. Let P and Q be points on the curve with chord PQ parallel to the tangent at T (Fig. 3). Extend PT to P' with $P'T = PT$. Locate Q' similarly. Then $PQP'Q'$ is a parallelogram with PQ and $P'Q'$ equally distant from T. When this construction is repeated for every line PQ, parallel to the tangent at T, the locus of P' and Q' is a parabola congruent and symmetrical to the first through T, and with the midpoints of the parallel chords collinear on a diameter of the curves. This new curve cuts in real points all lines parallel to tangent at T, that the first did not, and conversely.

To solve graphically for imaginary points of intersection of a line and parabola: draw two chords parallel to the line and join their midpoints obtaining a diameter to cut curve in T and l in M. On MT take $TO = MT$ and through O draw PQ parallel to l. Then PT cuts l in P' and QT cuts it in Q', and the coördinates of the imaginary points of intersection are read out as before.

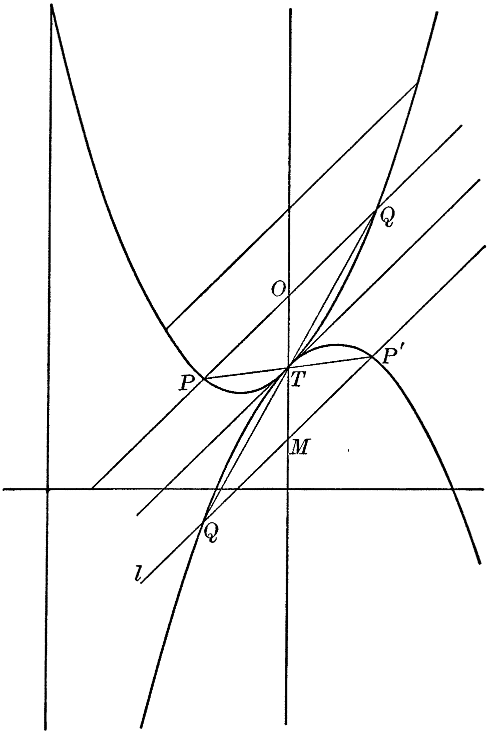

FIG. 3.

Solution of a quadratic equation.—Two commonly used graphical solutions of a quadratic equation

$$x^2 + px + q = 0 \tag{1}$$

depend on parabolas. In the first we plot the function

$$y = x^2 + px + q \tag{2}$$

and read out the intercepts on the x-axis. The failure of this method when the roots are complex can be remedied by drawing a line parallel to the x-axis with the vertex of the curve half way between them and proceeding as in the previous paragraph.

In the second method, we remark that the function (2) can be regarded as the sum of the expressions

$$y_1 = x^2 \qquad \text{and} \qquad y_2 = -px - q.$$

Thus the parabola $y_1 = x^2$ will be the same for all quadratics and is graphed once for all. Each of the two points in which the line $y_2 = -px - q$ cuts the curve has an abscissa x which makes $y_1 = y_2$ and so is a root of (1). With the addition of the methods of the preceding section we have a construction that solves every quadratic equation graphically.

Note: The topic of this paper is discussed in J. G. Hamilton and F. Kettle, *Graphs and Imaginaries*, London, Arnold, 1904; and A. Schultze, *Graphic Algebra*, New York, Macmillan, 1908.—EDITOR-IN-CHIEF.